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Acoustic radiation pressure on a compressible sphere in a viscous fluid

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The acoustic radiation pressure exerted by a plane – progressive or standing – sound wave on a compressible sphere suspended freely in a viscous fluid is calculated. In deriving the general expression for the radiation pressure, it is supposed that the radius of the sphere is arbitrary. Two limiting cases of interest are then considered. In the first of these, it is assumed that the sound wavelength is much larger than the radius of the sphere which is, in turn, much larger than the viscous wavelength, it being supposed that this condition is satisfied both outside and inside the sphere. In the second case, the situation is investigated when the radius of the sphere is small compared with the viscous wavelength which is, in turn, much smaller than the sound wavelength, it being supposed that this condition is satisfied, as before, both outside and inside the sphere. It is shown that in both cases the expressions for the radiation pressure are drastically different from the well-known expressions for the radiation pressure in a perfect fluid: the calculation of the radiation pressure from the formulae obtained for a perfect fluid in the cases when the effect of viscosity is not negligible gives both quantitatively and qualitatively wrong results.

1. Introduction

An object placed in a sound field is known to experience a steady force which is usually called the acoustic radiation pressure. This force is an analogue of the optical radiation pressure exerted by an electromagnetic wave on an electrically or magnetically responsive object, but the radiation force generated by a sound field is in general substantially larger than the electromagnetic radiation force (Jones & Leslie 1978).

For this reason, the acoustic radiation pressure is found useful in many applications, such as acoustic levitation, medical ultrasound equipment, etc. (Wu & Du 1990). For example, in acoustic levitation the radiation pressure is used to counteract the gravitational field for the purpose of containerless processing of materials and studies in fluid dynamics (Trinh 1985; Lee, Anilkumar & Wang 1991). This force also plays an important role in other acoustic phenomena, such as acoustic cavitation and sonoluminescence (Walton & Reynolds 1984).

The original theory of acoustic radiation pressure was proposed by Rayleigh (1894, pp. 43–45). The acoustic radiation pressure on a rigid sphere suspended freely in a plane progressive sound wave field or plane standing sound wave field was first calculated by King (1934), who, however, considered the radius of the sphere to be arbitrary. The compressibility of the sphere was taken into account by Yosioka & Kawasima (1955). This allowed King's theory to be generalized to cases of dispersed particles of finite compressibility, such as the motion of gas bubbles in a liquid or of liquid drops in another liquid.

However, the above authors and others who have dealt with this matter have neglected the effects of viscosity: as far as we are aware there are no works in which these effects have been taken into account consistently and rigorously. In the present paper this gap in the theory is filled, i.e. the acoustic radiation pressure on the sphere is calculated assuming that the media outside and inside the sphere behave as viscous compressible barotropic fluids. The theoretical investigation involves the calculation of the radiation pressure exerted by a plane – progressive or standing – sound wave on the sphere, it being supposed that the radius of the sphere is arbitrary. The limiting cases are then considered when $\lambda_v \ll R \ll \lambda_s$, $\tilde{\lambda}_v \ll R \ll \tilde{\lambda}_s$ or $R \ll \lambda_v \ll \lambda_s$, $R \ll \tilde{\lambda}_v \ll \tilde{\lambda}_s$, where R is the radius of the sphere; λ_s, $\tilde{\lambda}_s$ are the sound wavelengths outside and inside the sphere, respectively; and λ_v, $\tilde{\lambda}_v$ are the viscous wavelengths outside and inside the sphere, respectively. To anticipate, the expression for the radiation pressure in a viscous fluid is found to be drastically different from that in a perfect fluid. In particular, in a viscous fluid the radiation pressure due to a plane progressive wave can cause the sphere to move both in the direction of wave propagation and in the opposite direction, while in a perfect fluid the sphere is always urged away in the direction of wave propagation (Yosioka & Kawasima 1955). These interesting phenomena are also observed in a standing wave field.

2. Derivation of a general expression for the radiation pressure

Consider a spherical particle suspended freely in a viscous fluid whose motion is described by the following tensor equations:

$$\frac{\partial}{\partial t}(\rho v_i) = \frac{\partial}{\partial x_k}(\sigma_{ik} - \rho v_i v_k), \tag{2.1}$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_i}(\rho v_i), \tag{2.2}$$

$$\sigma_{ik} = -p\delta_{ik} + \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\frac{\partial v_j}{\partial x_j}\delta_{ik}\right) + \xi\frac{\partial v_j}{\partial x_j}\delta_{ik}, \tag{2.3}$$

$$p = p(\rho), \tag{2.4}$$

where σ_{ik} is the stress tensor, $\boldsymbol{v}$ is the fluid velocity, p is the fluid pressure, ρ is the fluid density, η is the shear viscosity, and ξ is the bulk viscosity. By 'spherical particle' we shall mean either a gas bubble in a liquid, or a liquid drop in a gas or in another liquid. So, the fluid surrounding the spherical particle can be either a liquid or a gas. This is also the case for the medium inside the sphere; we shall suppose that the medium inside the sphere behaves as a viscous fluid (liquid or gas) too, so that its motion is described by equations corresponding to (2.1)–(2.4).

As the sound wave propagates, the sphere will experience acoustic radiation pressure, written as follows:

$$F_i = \left\langle \int_{s(t)} \sigma_{ik} n_k \, \mathrm{d}s \right\rangle, \tag{2.5}$$

where $s(t)$ is the position of the sphere surface at time t, $\boldsymbol{n}$ is the outward normal to the sphere surface, and $\langle\ \rangle$ means an average over a sound wave cycle.

With accuracy up to the second-order terms in the amplitude of the incident sound wave, (2.5) can be rewritten as

$$F_i \approx \left\langle \int_{s(t)} \sigma_{ik}^{(1)} n_k \, \mathrm{d}s \right\rangle + \int_{s_0} \langle \sigma_{ik}^{(2)} \rangle n_k \, \mathrm{d}s, \tag{2.6}$$

where s_0 is the unperturbed surface of the sphere, $\sigma_{ik}^{(1)}$ is the stress tensor of first order in the sound wave amplitude, and $\sigma_{ik}^{(2)}$ is the stress tensor of second order.

The tensor $\sigma_{ik}^{(1)}$ satisfies the linearized equations (2.1)–(2.4) which have the form

$$\rho_0 \frac{\partial v_i^{(1)}}{\partial t} = \frac{\partial \sigma_{ik}^{(1)}}{\partial x_k}, \tag{2.7}$$

$$\frac{\partial \rho^{(1)}}{\partial t} = -\rho_0 \frac{\partial v_i^{(1)}}{\partial x_i}, \tag{2.8}$$

$$p^{(1)} = c^2 \rho^{(1)}, \tag{2.9}$$

where ρ_0 is the equilibrium fluid density, and c is the sound speed. Here, we have used the superscript (1) to denote quantities of first order in the sound wave amplitude.

The tensor $\langle \sigma_{ik}^{(2)} \rangle$ in a perfect fluid is known to be expressed through terms of first order only. This is not the case in a viscous fluid. In order to find $\langle \sigma_{ik}^{(2)} \rangle$ in a viscous fluid, one must solve the time-averaged viscous equations of motion in the second approximation, the so-called equations of acoustic streaming (Lighthill 1978):

$$\frac{\partial}{\partial x_k} \langle \sigma_{ik}^{(2)} \rangle = \rho_0 \frac{\partial}{\partial x_k} \langle v_i^{(1)} v_k^{(1)} \rangle, \tag{2.10}$$

$$\frac{\partial}{\partial x_i} \langle v_i^{(2)} \rangle = -\frac{1}{\rho_0} \frac{\partial}{\partial x_i} \langle \rho^{(1)} v_i^{(1)} \rangle, \tag{2.11}$$

where $\boldsymbol{v}^{(2)}$ is the fluid velocity of second order.

We shall return to (2.7)–(2.11) below, but now let us transform the first term on the right-hand side of (2.6). It can be represented as

$$\left\langle \int_{s(t)} \sigma_{ik}^{(1)} n_k \, \mathrm{d}s \right\rangle = \left\langle \int_{\bar{s}} \sigma_{ik}^{(1)} n_k \, \mathrm{d}s \right\rangle - \left\langle \int_{\tau(t)} \frac{\partial \sigma_{ik}^{(1)}}{\partial x_k} \mathrm{d}\tau \right\rangle, \tag{2.12}$$

where $\bar{s}$ is a fixed surface surrounding the sphere, and $\tau(t)$ is the volume bounded by the surfaces $s(t)$ and $\bar{s}$.

The first term on the right-hand side of (2.12) is equal to zero. By using (2.7) and the relation (Yosioka & Kawasima 1955)

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\tau(t)} \boldsymbol{v}^{(1)} \mathrm{d}\tau = \int_{\tau(t)} \frac{\partial \boldsymbol{v}^{(1)}}{\partial t} \mathrm{d}\tau - \int_{s(t)} \boldsymbol{v}^{(1)} (\boldsymbol{v}^{(1)} \cdot \boldsymbol{n}) \, \mathrm{d}s, \tag{2.13}$$

one obtains

$$\left\langle \int_{s(t)} \sigma_{ik}^{(1)} n_k \, \mathrm{d}s \right\rangle = -\rho_0 \int_{\bar{s}} \langle v_i^{(1)} v_k^{(1)} \rangle n_k \, \mathrm{d}s. \tag{2.14}$$

Here, we have also used the fact that

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t} \int_{\tau(t)} \boldsymbol{v}^{(1)} \mathrm{d}\tau \right\rangle = 0. \tag{2.15}$$

Substitution of (2.14) into (2.6) yields

$$F_i = \int_{s_0} \langle \sigma_{ik}^{(2)} - \rho_0 v_i^{(1)} v_k^{(1)} \rangle n_k \, \mathrm{d}s. \tag{2.16}$$

It follows from (2.10) that the integral in (2.16) may be taken over any surface surrounding the sphere. By using (2.3) and the relation

$$n_k\left(\frac{\partial v_k^{(2)}}{\partial x_i}-\frac{\partial v_i^{(2)}}{\partial x_k}\right) = (\boldsymbol{n}\times(\boldsymbol{\nabla}\times \boldsymbol{v}^{(2)}))_i, \tag{2.17}$$

we can write (2.16) in the vector form

$$\boldsymbol{F} = \int_{s_0} \langle 2\eta(\boldsymbol{n}\cdot\boldsymbol{\nabla})\,\boldsymbol{v}^{(2)}+\eta\boldsymbol{n}\times(\boldsymbol{\nabla}\times\boldsymbol{v}^{(2)})+(\xi-\tfrac{2}{3}\eta)\,\boldsymbol{n}(\boldsymbol{\nabla}\cdot\boldsymbol{v}^{(2)})-\boldsymbol{n}p^{(2)}-\rho_0\,\boldsymbol{v}^{(1)}(\boldsymbol{v}^{(1)}\cdot\boldsymbol{n})\rangle\,\mathrm{d}s, \tag{2.18}$$

where $p^{(2)}$ is the fluid pressure of second order.

In order to go on with the calculation of the radiation pressure, one must solve (2.7)–(2.11). The next two sections are devoted to finding these solutions.

3. Solution of the linearized viscous equations of motion

The velocity $\boldsymbol{v}^{(1)}$ can be represented as (Lamb 1932)

$$\boldsymbol{v}^{(1)} = \boldsymbol{\nabla}\phi+\boldsymbol{\nabla}\times\boldsymbol{\psi}, \tag{3.1}$$

where ϕ and $\boldsymbol{\psi}$ are the scalar and vorticity velocity potentials in the first approximation. As usual, the scalar potential can be written as follows:

$$\phi = \phi_I+\phi_S, \tag{3.2}$$

where ϕ_I is the scalar velocity potential of the incident sound wave, and ϕ_S is the first-order scalar velocity potential of the scattered wave.

Note that for progressive and standing waves, the problem involved is axisymmetric. Let the origin of the spherical coordinate system (r, θ, ϵ) be at the equilibrium centre of the sphere and the wavevector $\boldsymbol{k}$ be in the direction of the z-axis. Then the scalar potential ϕ_I, in the general case, can be written as

$$\phi_I = \exp(-\mathrm{i}\omega t)\sum_{n=0}^{\infty} A_n j_n(kr)\,P_n(\cos\theta), \tag{3.3}$$

where ω is the angular frequency of the incident sound wave, k is the wavenumber, given by

$$k = \frac{\omega}{c}\left[1-\frac{i\omega}{\rho_0 c^2}(\xi+\tfrac{4}{3}\eta)\right]^{-\frac{1}{2}}, \tag{3.4}$$

j_n is the spherical Bessel function of order n, P_n is the Legendre polynomial of degree n, and the A_n are constants to be determined by the type of incident wave.

The potentials ϕ_S and $\boldsymbol{\psi}$ are found from (2.7) and (2.8) which, by means of (2.3), (2.9) and (3.1)–(3.3), are reduced to the form

$$(\Delta+k^2)\,\phi_S = 0, \tag{3.5}$$

$$(\Delta+k_v^2)\,\boldsymbol{\psi} = 0, \tag{3.6}$$

where $k_v = (1+\mathrm{i})/\delta$, $\delta = (2\nu/\omega)^{\frac{1}{2}}$, $\nu = \eta/\rho_0$, ν is the kinematic viscosity, and the quantity $\lambda_v = 2\pi\delta$ is usually called the viscous wavelength.

By using (2.8) and (3.1)–(3.5), from (2.9) one obtains an expression for $p^{(1)}$ which is also needed for the subsequent calculations:

$$p^{(1)} = \mathrm{i}\rho_0\,c^2k^2\phi/\omega, \tag{3.7}$$

The axisymmetric solutions of (3.5) and (3.6) meeting Sommerfeld's radiation conditions at infinity are given by

$$\phi_S = \exp(-\mathrm{i}\omega t) \sum_{n=0}^{\infty} \alpha_n A_n h_n(kr) P_n(\cos\theta), \tag{3.8}$$

$$\boldsymbol{\psi} = \exp(-\mathrm{i}\omega t)\, \boldsymbol{e}_\epsilon \sum_{n=1}^{\infty} \beta_n A_n h_n(k_v r) P_n^{(1)}(\cos\theta), \tag{3.9}$$

where h_n is the spherical Hankel function of the first kind, $P_n^{(1)}$ is the associated Legendre polynomial of the first order, $\boldsymbol{e}_\epsilon$ is the unit vector of the spherical coordinate system, and α_n and β_n are constants to be determined by the boundary conditions at the sphere surface.

Like $\boldsymbol{v}^{(1)}$, the first-order velocity of the medium inside the sphere can be represented as

$$\tilde{\boldsymbol{v}}^{(1)} = \nabla\tilde{\phi} + \nabla \times \tilde{\boldsymbol{\psi}}, \tag{3.10}$$

where the tilde denotes quantities that concern the medium inside the sphere.

There is only a refracted wave inside the sphere and its amplitude must be finite at $r = 0$. Therefore, instead of (3.8) and (3.9) we have

$$\tilde{\phi} = \exp(-\mathrm{i}\omega t) \sum_{n=0}^{\infty} \tilde{\alpha}_n A_n j_n(\tilde{k}r) P_n(\cos\theta), \tag{3.11}$$

$$\tilde{\boldsymbol{\psi}} = \exp(-\mathrm{i}\omega t)\, \boldsymbol{e}_\epsilon \sum_{n=1}^{\infty} \tilde{\beta}_n A_n j_n(\tilde{k}_v r) P_n^{(1)}(\cos\theta), \tag{3.12}$$

where $\tilde{\alpha}_n$ and $\tilde{\beta}_n$, like α_n and β_n, are constants to be determined by the boundary conditions at the sphere surface.

The above boundary conditions are expressed as follows:

$$\boldsymbol{v}^{(1)} = \tilde{\boldsymbol{v}}^{(1)} \quad \text{at} \quad r = R, \tag{3.13}$$

$$(\sigma_{ik}^{(1)} - (p_0 + p_{ST})\,\delta_{ik})\, n_k = (\tilde{\sigma}_{ik}^{(1)} - \tilde{p}_0\,\delta_{ik})\, n_k \quad \text{at} \quad r = R, \tag{3.14}$$

where p_0 and $\tilde{p}_0$ are the equilibrium pressures outside and inside the sphere, respectively, and p_{ST} is the pressure of the surface tension.

In spherical coordinates (3.13) and (3.14) take the form

$$v_r^{(1)} = \tilde{v}_r^{(1)}, \quad v_\theta^{(1)} = \tilde{v}_\theta^{(1)} \quad \text{at} \quad r = R, \tag{3.15}$$

$$\sigma_{rr}^{(1)} - p_{ST} - p_0 = \tilde{\sigma}_{rr}^{(1)} - \tilde{p}_0, \quad \sigma_{r\theta}^{(1)} = \tilde{\sigma}_{r\theta}^{(1)} \quad \text{at} \quad r = R. \tag{3.16}$$

By using the formulae for the divergences and the curls of the velocities $\boldsymbol{v}^{(1)}$ and $\tilde{\boldsymbol{v}}^{(1)}$ in spherical coordinates and taking into account (3.15), one can rewrite (3.16) as follows:

$$\phi - \left(\frac{\tilde{\rho}_0}{\rho_0}\right)\tilde{\phi} + \frac{2R}{x_v^2}\left(1 - \frac{\tilde{\eta}}{\eta}\right)\left(2\tilde{v}_r^{(1)} + \operatorname{cotan}(\theta)\,\tilde{v}_\theta^{(1)} + \frac{\partial \tilde{v}_\theta^{(1)}}{\partial\theta}\right) = \frac{\tilde{p}_0 - p_0 - p_{ST}}{\mathrm{i}\omega\rho_0} \quad \text{at} \quad r = R, \tag{3.17}$$

$$\boldsymbol{\psi}\cdot\boldsymbol{e}_\epsilon - \left(\frac{\tilde{\rho}_0}{\rho_0}\right)\tilde{\boldsymbol{\psi}}\cdot\boldsymbol{e}_\epsilon + \frac{2R}{x_v^2}\left(1 - \frac{\tilde{\eta}}{\eta}\right)\left(\frac{\partial \tilde{v}_r^{(1)}}{\partial\theta} - \tilde{v}_\theta^{(1)}\right) = 0 \quad \text{at} \quad r = R, \tag{3.18}$$

where $x_v = k_v R$. These equations are more convenient for the subsequent calculations than (3.16).

The pressure of the surface tension is known to be given by

$$p_{ST} = -2\sigma H, \tag{3.19}$$

where σ is the surface tension coefficient, and H is the mean surface curvature. The formula for H can be found, for example, in Korn & Korn (1968). However, to calculate H by this formula, an expression is needed for the perturbed surface of the sphere. This expression can be written as

$$r = R + \zeta(\theta)\exp(-\mathrm{i}\omega t), \tag{3.20}$$

the function $\zeta(\theta)$ being given by

$$\zeta(\theta)\exp(-\mathrm{i}\omega t) = \int \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}t = -\dot{\boldsymbol{u}}\cdot\boldsymbol{n}/\omega^2, \tag{3.21}$$

where $\boldsymbol{u}$ is the sphere surface velocity of first order, $\dot{\boldsymbol{u}} = \mathrm{d}\boldsymbol{u}/\mathrm{d}t$, and the second equality in (3.21) follows from the fact that $\boldsymbol{u} \sim \exp(-\mathrm{i}\omega t)$.

Clearly

$$\boldsymbol{u} = \tilde{\boldsymbol{v}}^{(1)} = \boldsymbol{v}^{(1)} \quad \text{at} \quad r = R. \tag{3.22}$$

Replacing $\boldsymbol{u}$ in (3.21) by $\tilde{\boldsymbol{v}}^{(1)}$ at $r = R$, substituting (3.31) into (3.20), and (3.20) into the formula for H (see Korn & Korn 1968), one obtains an expression for H accurate to the first-order terms in the amplitude of the incident sound wave as

$$H = H_0 + H^{(1)}, \tag{3.23}$$

where

$$H_0 = -\frac{1}{R}, \quad H^{(1)} = -\exp(-\mathrm{i}\omega t)\sum_{n=0}^{\infty}\frac{(n-1)(n+2)}{2R^2}\zeta_n(\theta), \tag{3.24}$$

$$\zeta_n(\theta) = \frac{\mathrm{i}A_n}{\omega R}[\tilde{x}j_n'(\tilde{x})\,\tilde{\alpha}_n + n(n+1)j_n(\tilde{x}_v)\,\tilde{\beta}_n]\,P_n(\cos\theta), \tag{3.25}$$

$\tilde{x} = \tilde{k}R$, $\tilde{x}_v = \tilde{k}_v R$, and $j_n'(\tilde{x}) = \mathrm{d}j_n(\tilde{x})/\tilde{x}$.

Substituting (3.23)–(3.25) into (3.19), we can find p_{ST}. Thus, we have all the necessary quantities for calculating the constants α_n, β_n, $\tilde{\alpha}_n$ and $\tilde{\beta}_n$. Substituting (3.1)–(3.3), (3.8)–(3.12), (3.19) and (3.23)–(3.25) into (3.15), (3.17) and (3.18), one obtains four combined algebraic equations in the unknowns α_n, β_n, $\tilde{\alpha}_n$ and $\tilde{\beta}_n$. These equations can be written in matrix form as

$$\boldsymbol{M}\cdot\boldsymbol{X} = \boldsymbol{N}, \tag{3.26}$$

where $\boldsymbol{X}$ and $\boldsymbol{N}$ are given by

$$\boldsymbol{X} = \begin{bmatrix}\alpha_n\\ \beta_n\\ \tilde{\alpha}_n\\ \tilde{\beta}_n\end{bmatrix}, \quad \boldsymbol{N} = \begin{bmatrix}xj_n'(x)\\ j_n(x)\\ j_n(x)\\ 0\end{bmatrix}, \tag{3.27}$$

in which $x = kR$. The elements of the matrix $\boldsymbol{M}$, for which $n \geqslant 1$, are given in Appendix A. For $n = 0$, equations (3.26) are reduced to two combined equations in the unknowns α_0 and $\tilde{\alpha}_0$ ($\beta_0 = \tilde{\beta}_0 = 0$). By solving them, one obtains

$$\alpha_0 = \frac{xj_1(x) - bj_0(x)}{bh_0(x) - xh_1(x)}, \tag{3.28}$$

where

$$b = \tilde{x}j_1(\tilde{x})\left\{\frac{\tilde{\rho}_0}{\rho_0}j_0(\tilde{x}) + 2\tilde{x}j_1(\tilde{x})\left[\frac{2(1-\tilde{\eta}/\eta)}{x_v^2} - \frac{\sigma}{\rho_0\,\omega^2R^3}\right]\right\}^{-1} \tag{3.29}$$

(the constant $\tilde{\alpha}_0$ is unnecessary for the calculation of the radiation pressure).

For $n \geqslant 1$, equations (3.26) can be easily solved by one of the conventional methods. However, the exact expressions for α_n and β_n are very complicated. Therefore, they are not given in this paper. The approximate expressions for α_n and β_n ($n = 0, 1, 2$) in some limiting cases of interest are given in Appendix B. We shall consider these cases later.

4. Solution of the equations of acoustic streaming

The velocity $\langle \boldsymbol{v}^{(2)} \rangle$ can be represented as follows:

$$\langle \boldsymbol{v}^{(2)} \rangle = \langle \boldsymbol{v}_I^{(2)} \rangle + \langle \boldsymbol{v}_S^{(2)} \rangle, \tag{4.1}$$

where $\langle \boldsymbol{v}_I^{(2)} \rangle$ is the velocity of the stationary flow which exists in the sound field in absence of the sphere, and $\langle \boldsymbol{v}_S^{(2)} \rangle$ is the velocity of the acoustic streaming arising around the sphere. Like $\boldsymbol{v}^{(1)}$, the velocity $\langle \boldsymbol{v}_S^{(2)} \rangle$ can be written as

$$\langle \boldsymbol{v}_S^{(2)} \rangle = \nabla \Phi + \nabla \times \boldsymbol{\Psi}, \tag{4.2}$$

where Φ and $\boldsymbol{\Psi}$ are the scalar and vorticity velocity potentials of the acoustic streaming.

Substituting (4.2) into (2.3), (2.10) and (2.11) we obtain, after some manipulation, equations for Φ, $\boldsymbol{\Psi}$ and $\langle p_S^{(2)} \rangle$ (we write $\langle p_S^{(2)} \rangle$ for the pressure of the acoustic streaming) as

$$\Delta \Phi = -\rho_0^{-1} \nabla \cdot \langle \rho^{(1)} \boldsymbol{v}^{(1)} \rangle_S, \tag{4.3}$$

$$\Delta \Delta \boldsymbol{\Psi} = -\nu^{-1} \nabla \times \langle \boldsymbol{v}^{(1)} (\nabla \cdot \boldsymbol{v}^{(1)}) + (\boldsymbol{v}^{(1)} \cdot \nabla)\, \boldsymbol{v}^{(1)} \rangle_S, \tag{4.4}$$

$$\nabla \langle p_S^{(2)} \rangle = \eta \Delta \nabla \times \boldsymbol{\Psi} - \rho_0 \langle \boldsymbol{v}^{(1)} (\nabla \cdot \boldsymbol{v}^{(1)}) + (\boldsymbol{v}^{(1)} \cdot \nabla)\, \boldsymbol{v}^{(1)} \rangle_S - \rho_0^{-1} (\xi + \tfrac{4}{3}\eta)\, \nabla (\nabla \cdot \langle \rho^{(1)} \boldsymbol{v}^{(1)} \rangle)_S. \tag{4.5}$$

Here, we have also used the subscript S on the right-hand sides to indicate that the expressions do not involve components that are only expressed in terms of the incident wave. For example, $\langle \rho^{(1)} \boldsymbol{v}^{(1)} \rangle_S = \langle \rho^{(1)} \boldsymbol{v}^{(1)} \rangle - \langle \rho_I^{(1)} \boldsymbol{v}_I^{(1)} \rangle$, where $\boldsymbol{v}_I^{(1)} = \nabla \phi_I$ and $\rho_I^{(1)} = \mathrm{i} \rho_0 k^2 \phi_I / \omega$ (see (2.9) and (3.7)).

First, we shall solve (4.3). The term on its right-hand side can be expanded in Legendre polynomials as

$$\rho_0^{-1} \nabla \cdot \langle \rho^{(1)} \boldsymbol{v}^{(1)} \rangle_S = \sum_{n=0}^{\infty} \frac{(2n+1)}{R^2} \mu_n(y)\, P_n(\cos\theta), \tag{4.6}$$

where

$$\mu_n(y) = \frac{R^2}{2\rho_0} \int_0^{\pi} \nabla \cdot \langle \rho^{(1)} \boldsymbol{v}^{(1)} \rangle_S\, P_n(\cos\theta) \sin\theta \,\mathrm{d}\theta, \tag{4.7}$$

and $y = r/R$.

It is obvious that Φ should be sought in the following form:

$$\Phi = \sum_{n=0}^{\infty} \Phi_n(y)\, P_n(\cos\theta). \tag{4.8}$$

Substituting (4.6) and (4.8) into (4.3), one obtains an equation for the function $\Phi_n(y)$:

$$\Phi_n''(y) + \frac{2}{y} \Phi_n'(y) - \frac{n(n+1)}{y^2} \Phi_n(y) = -(2n+1)\, \mu_n(y), \tag{4.9}$$

where $\Phi_n'(y) = \mathrm{d}\Phi_n(y)/\mathrm{d}y$.

The solution of (4.9) is

$$\Phi_n(y) = y^{-(n+1)}\left(\int_1^y z^{n+2}\mu_n(z)\,\mathrm{d}z - C_{1n}\right) - y^n\left(\int_1^y z^{-(n-1)}\mu_n(z)\,\mathrm{d}z - C_{2n}\right). \quad (4.10)$$

It follows from the condition $\nabla\Phi \to 0$ as $r \to \infty$ that

$$C_{2n} = \int_1^\infty z^{-(n-1)}\mu_n(z)\,\mathrm{d}z. \quad (4.11)$$

The constant C_{1n} is to be determined by the boundary conditions at the sphere surface.

The solution of (4.3) has been obtained, and now we look for the solution of (4.4). The term on its right-hand side can be expanded in Legendre polynomials and in associated Legendre polynomials as

$$\nu^{-1}\langle \boldsymbol{v}^{(1)}(\nabla\cdot\boldsymbol{v}^{(1)}) + (\boldsymbol{v}^{(1)}\cdot\nabla)\,\boldsymbol{v}^{(1)}\rangle_S = \frac{\boldsymbol{e}_r}{R^3}\sum_{n=0}^{\infty}(2n+1)\,\chi_{rn}(y)\,P_n(\cos\theta)$$
$$-\frac{\boldsymbol{e}_\theta}{R^3}\sum_{n=1}^{\infty}(2n+1)\,\chi_{\theta n}(y)\,P_n^{(1)}(\cos\theta), \quad (4.12)$$

where

$$\chi_{rn}(y) = \frac{R^3}{2\nu}\int_0^\pi \boldsymbol{e}_r\cdot\langle \boldsymbol{v}^{(1)}(\nabla\cdot\boldsymbol{v}^{(1)}) + (\boldsymbol{v}^{(1)}\cdot\nabla)\,\boldsymbol{v}^{(1)}\rangle_S\, P_n(\cos\theta)\sin\theta\,\mathrm{d}\theta, \quad (4.13)$$

$$\chi_{\theta n}(y) = -\frac{R^3}{2\nu n(n+1)}\int_0^\pi \boldsymbol{e}_\theta\cdot\langle \boldsymbol{v}^{(1)}(\nabla\cdot\boldsymbol{v}^{(1)}) + (\boldsymbol{v}^{(1)}\cdot\nabla)\,\boldsymbol{v}^{(1)}\rangle_S\, P_n^{(1)}(\cos\theta)\sin\theta\,\mathrm{d}\theta. \quad (4.14)$$

This term can be written as

$$\nu^{-1}\langle \boldsymbol{v}^{(1)}(\nabla\cdot\boldsymbol{v}^{(1)}) + (\boldsymbol{v}^{(1)}\cdot\nabla)\,\boldsymbol{v}^{(1)}\rangle_S = \nabla Q + \nabla\times\boldsymbol{q}, \quad (4.15)$$

where

$$Q = R^{-2}\sum_{n=0}^{\infty} Q_n(y)\,P_n(\cos\theta), \quad (4.16)$$

$$\boldsymbol{q} = \boldsymbol{e}_\epsilon R^{-2}\sum_{n=1}^{\infty} q_n(y)\,P_n^{(1)}(\cos\theta). \quad (4.17)$$

Equating the terms on the right-hand side of (4.15) to those on the right-hand side of (4.12), one obtains two simultaneous equations for the functions $Q_n(y)$ and $q_n(y)$:

$$yQ_n'(y) + n(n+1)\,q_n(y) = (2n+1)\,y\chi_{rn}(y), \quad (4.18)$$
$$Q_n(y) + yq_n'(y) + q_n(y) = (2n+1)\,y\chi_{\theta n}(y). \quad (4.19)$$

Solving (4.18) and (4.19) by the Lagrange method, one finds

$$Q_n(y) = y^{-(n+1)}\left(n\int_1^y z^{n+1}[\chi_{rn}(z) + (n+1)\,\chi_{\theta n}(z)]\,\mathrm{d}z - C_{3n}\right)$$
$$+ y^n\left((n+1)\int_1^y z^{-n}[\chi_{rn}(z) - n\chi_{\theta n}(z)]\,\mathrm{d}z - C_{4n}\right), \quad (4.20)$$

$$q_n(y) = y^{-(n+1)}\left(\int_1^y z^{n+1}[\chi_{rn}(z) + (n+1)\,\chi_{\theta n}(z)]\,\mathrm{d}z - \frac{C_{3n}}{n}\right)$$
$$- y^n\left(\int_1^y z^{-n}[\chi_{rn}(z) - n\chi_{\theta n}(z)]\,\mathrm{d}z - \frac{C_{4n}}{n+1}\right). \quad (4.21)$$

It follows from the boundary conditions at infinity that

$$C_{4n} = (n+1)\int_1^\infty z^{-n}\,[\chi_{rn}(z) - n\chi_{\theta n}(z)]\,\mathrm{d}z. \tag{4.22}$$

Like C_{1n}, the constant C_{3n} is to be determined by the boundary conditions at the sphere surface.

Further, substitution of (4.15) into (4.4) yields

$$\Delta\Delta\boldsymbol{\Psi} = \Delta\boldsymbol{q} \tag{4.23}$$

or equivalently

$$\Delta\boldsymbol{\Psi} = \boldsymbol{q}. \tag{4.24}$$

Taking into account (4.17), the solution of (4.24) is sought in the form

$$\boldsymbol{\Psi} = \boldsymbol{e}_\epsilon \sum_{n=1}^{\infty} \Psi_n(y)\, P_n^{(1)}(\cos\theta), \tag{4.25}$$

Substituting (4.25) into (4.24), one obtains an equation for the function $\Psi_n(y)$:

$$\Psi_n''(y) + \frac{2}{y}\Psi_n'(y) - \frac{n(n+1)}{y^2}\Psi_n(y) = q_n(y). \tag{4.26}$$

The calculations give

$$\begin{aligned}\Psi_n(y) = {} & \frac{y^{-(n+1)}}{2(2n+3)}\left(\int_1^y z^{n+3}\,[\chi_{rn}(z) + (n+3)\,\chi_{\theta n}(z)]\,\mathrm{d}z - C_{6n}\right)\\ & - \frac{y^{-(n-1)}}{2(2n-1)}\left(\int_1^y z^{n+1}\,[\chi_{rn}(z) + (n+1)\,\chi_{\theta n}(z)]\,\mathrm{d}z - \frac{C_{3n}}{n}\right)\\ & + \frac{y^{n}}{2(2n-1)}\left(\int_1^y z^{-(n-2)}\,[\chi_{rn}(z) - (n-2)\,\chi_{\theta n}(z)]\,\mathrm{d}z - C_{5n}\right)\\ & - \frac{y^{n+2}}{2(2n+3)}\left(\int_1^y z^{-n}\,[\chi_{rn}(z) - n\chi_{\theta n}(z)]\,\mathrm{d}z - \frac{C_{4n}}{n+1}\right).\end{aligned} \tag{4.27}$$

Note that (4.10) contains the term C_{1n}/y^{n+1} which yields the same velocity field as the term $C_{6n}/2(2n+3)\,y^{n+1}$ of (4.27).

Actually,

$$\nabla\times[\boldsymbol{e}_\epsilon P_n^{(1)}(\cos\theta)/y^{n+1}] = -n\nabla[P_n(\cos\theta)/y^{n+1}]. \tag{4.28}$$

In view of this, the above-mentioned term in (4.27) can be dropped, i.e. we can assume $C_{6n} = 0$.

It follows from the condition $\nabla\times\boldsymbol{\Psi}\to 0$ as $r\to\infty$ that

$$C_{5n} = \int_1^y z^{-(n-2)}\,[\chi_{rn}(z) - (n-2)\,\chi_{\theta n}(z)]\,\mathrm{d}z. \tag{4.29}$$

Finally, substituting (4.15) into (4.5) and using (4.24), one obtains an expression for $\langle p_S^{(2)}\rangle$ as

$$\langle p_S^{(2)}\rangle = -\eta Q - \rho_0^{-1}(\xi + 4\eta/3)\,\nabla\cdot\langle\rho^{(1)}\boldsymbol{v}^{(1)}\rangle_S. \tag{4.30}$$

With the exception of the constants C_{1n} and C_{3n}, the solutions of the equations of acoustic streaming outside the sphere have been derived. To find the above constants, the following boundary condition should be applied

$$\boldsymbol{v} = \tilde{\boldsymbol{v}} \quad \text{at} \quad r = R + \zeta(\theta)\exp(-\mathrm{i}\omega t), \tag{4.31}$$

where $\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$ are the exact velocities of the media outside and inside the sphere, respectively. That is to say, we apply the boundary condition of continuity of the fluid velocity across the sphere surface to the perturbed surface of the sphere. This boundary condition should be written with an accuracy up to the second-order terms and be averaged over a sound wave cycle. This results in

$$\langle \boldsymbol{v}^{(2)} \rangle - \omega^{-2} \left\langle \dot{u}_r \frac{\partial \boldsymbol{v}^{(1)}}{\partial r} \right\rangle = \langle \tilde{\boldsymbol{v}}^{(2)} \rangle - \omega^{-2} \left\langle \dot{u}_r \frac{\partial \tilde{\boldsymbol{v}}^{(1)}}{\partial r} \right\rangle \quad \text{at} \quad r = R. \tag{4.32}$$

Suppose that the sphere does not execute a 'slow' motion. To put it in another way, we suppose that the sphere is not moving in the second approximation (for example, we can assume that the sphere is acted on by an external force that is equal and opposite to the acoustic radiation force). Equation (4.32) then takes the form

$$\langle \boldsymbol{v}_S^{(2)} \rangle = \omega^{-2} \left\langle \dot{u}_r \frac{\partial \boldsymbol{v}^{(1)}}{\partial r} \right\rangle - \langle \boldsymbol{v}_I^{(2)} \rangle \quad \text{at} \quad r = R. \tag{4.33}$$

Thus, for the calculation of the constants C_{1n} and C_{3n}, it is not necessary to solve the equations of acoustic streaming inside the sphere.

The terms on the right-hand side of (4.33) can be written as

$$\langle \boldsymbol{v}_I^{(2)} \rangle - \omega^{-2} \left\langle \dot{u}_r \frac{\partial \boldsymbol{v}^{(1)}}{\partial r} \right\rangle = \frac{\boldsymbol{e}_r}{R} \sum_{n=0}^{\infty} (2n+1)\, a_n P_n(\cos\theta)$$
$$- \frac{\boldsymbol{e}_\theta}{R} \sum_{n=1}^{\infty} (2n+1)\, b_n P_n^{(1)}(\cos\theta) \quad \text{at} \quad r = R, \tag{4.34}$$

where

$$a_n = \frac{R}{2} \int_0^\pi \left\langle v_{Ir}^{(2)} - \omega^{-2} \dot{u}_r \frac{\partial v_r^{(1)}}{\partial r} \right\rangle P_n(\cos\theta) \sin\theta \, \mathrm{d}\theta \quad \text{at} \quad r = R, \tag{4.35}$$

$$b_n = \frac{R}{2n(n+1)} \int_0^\pi \left\langle \omega^{-2} \dot{u}_r \frac{\partial v_\theta^{(1)}}{\partial r} - v_{I\theta}^{(2)} \right\rangle P_n^{(1)}(\cos\theta) \sin\theta \, \mathrm{d}\theta \quad \text{at} \quad r = R. \tag{4.36}$$

Substituting (4.2), (4.8), (4.10), (4.25), (4.27) and (4.34) into (4.33), one finds

$$C_1 = \frac{n(2n-1)}{2(n+1)} C_{2n} + \frac{n(2n+1)}{4(n+1)(2n+3)} C_{4n} - \tfrac{1}{4} n C_{5n}$$
$$+ \frac{(2n+1)(n-2)}{2(n+1)} a_n + \frac{n(2n+1)}{2} b_n, \quad n \geqslant 0, \tag{4.37}$$

$$C_{3n} = n(4n^2-1) \left[\frac{C_{5n}}{2(2n-1)} - \frac{C_{4n}}{2(n+1)(2n+1)} - \frac{C_{2n}}{n+1} - \frac{a_n}{n+1} - b_n \right], \quad n \geqslant 1. \tag{4.38}$$

5. Completion of the calculation of the radiation pressure

Now we can resume calculating the radiation pressure. Substituting (4.2), (4.6), (4.8), (4.25) and (4.30) into (2.18) and using (4.9), (4.24) and (4.26), one obtains

$$\boldsymbol{F} = -4\pi\eta C_{31} \boldsymbol{m} - \rho_0 \int_{S_0} \langle \boldsymbol{v}^{(1)} (\boldsymbol{v}^{(1)} \cdot \boldsymbol{n}) \rangle_S \, \mathrm{d}s, \tag{5.1}$$

where $\boldsymbol{m}$ is the unit vector in the direction of wave propagation. Then, substituting (4.11), (4.22) and (4.29) into (4.38), and (4.38), in its turn, into (5.1) and using the relation

$$2\pi\eta\boldsymbol{m}\int_1^\infty(3y-y^{-1})[\chi_{r1}(y)+2\chi_{\theta1}(y)]\,\mathrm{d}y$$
$$=\tfrac{3}{2}\rho_0\int_0^{2\pi}\mathrm{d}\epsilon\int_0^\pi\mathrm{d}\Omega\int_1^\infty\mathrm{d}y(1-y^{-2})\langle\boldsymbol{v}^{(1)}(\boldsymbol{v}^{(1)}\cdot\boldsymbol{n})\rangle_S-\rho_0\int_{s_0}\langle\boldsymbol{v}^{(1)}(\boldsymbol{v}^{(1)}\cdot\boldsymbol{n})\rangle_S\,\mathrm{d}s, \quad (5.2)$$

where $\mathrm{d}\Omega = R^2\sin\theta\,\mathrm{d}\theta$, one has

$$\boldsymbol{F} = 6\pi\eta\boldsymbol{m}(a_1+2b_1)+3\pi\nu\boldsymbol{m}\int_1^\infty\int_0^\pi\nabla\cdot\langle\rho^{(1)}\boldsymbol{v}^{(1)}\rangle_S\cos\theta\,\mathrm{d}\Omega\,\mathrm{d}y$$
$$-3\pi\rho_0\boldsymbol{m}\int_1^\infty\int_0^\pi(1-y^{-2})[\langle v_r^{(1)}v_r^{(1)}\cos\theta-v_r^{(1)}v_\theta^{(1)}\sin\theta\rangle_S$$
$$+\tfrac{1}{2}Ry\,\boldsymbol{e}_\theta\cdot\langle\boldsymbol{v}^{(1)}(\nabla\cdot\boldsymbol{v}^{(1)})+(\boldsymbol{v}^{(1)}\cdot\nabla)\,\boldsymbol{v}^{(1)}\rangle_S\sin\theta]\,\mathrm{d}\Omega\,\mathrm{d}y. \quad (5.3)$$

To complete the calculation of the radiation pressure, it is necessary to substitute the solutions of the linearized viscous equations of motion which have been derived in §3 into (5.3). After carrying out this operation, the general expression for the radiation pressure takes the final form

$$F = F_R+F_D, \quad (5.4)$$

where $F = \boldsymbol{F}\cdot\boldsymbol{m}$,

$$F_R = -\tfrac{3}{2}\pi\rho_0\sum_{n=0}^\infty\frac{n+1}{(2n+1)(2n+3)}(D_n\,A_n\,A_{n+1}^*+D_n^*\,A_n^*\,A_{n+1}), \quad (5.5)$$

$$D_n = S_{1n}\,\alpha_n\,\alpha_{n+1}^*+S_{2n}\,\beta_n\,\beta_{n+1}^*+S_{3n}\,\alpha_n\,\beta_{n+1}^*+S_{4n}\,\beta_n\,\alpha_{n+1}^*$$
$$+S_{5n}\,\alpha_n+S_{6n}\,\alpha_{n+1}^*+S_{7n}\,\beta_n+S_{8n}\,\beta_{n+1}^*+S_{9n}, \quad (5.6)$$

$$F_D = \frac{3\eta}{2R}\boldsymbol{m}\cdot\int_{s_0}\langle\boldsymbol{v}_I^{(2)}\rangle\,\mathrm{d}s. \quad (5.7)$$

The expressions for the functions S_{jn} ($j = 1, \ldots, 9$) are given in Appendix C.

The force F_R is just the radiation pressure. The force F_D is brought about by the stationary flow which is in the sound wave field even when the sphere is absent. This force is an analogue of Stokes' drag force.

We are interested in two types of sound waves: a plane progressive wave and a plane standing wave. The scalar velocity potential of the plane progressive wave is given by

$$\phi_I = A\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}-\mathrm{i}\omega t). \quad (5.8)$$

By expanding (5.8) in Legendre polynomials (Abramowitz & Stegun 1965) and comparing the series obtained with (3.3), one has for A_n

$$A_n = A(2n+1)\,\mathrm{i}^n. \quad (5.9)$$

Substituting (5.9) into (5.5), one obtains an expression for the radiation pressure exerted by the plane progressive wave:

$$F_R = \tfrac{3}{2}\mathrm{i}\pi\rho_0\,AA^*\sum_{n=0}^\infty(n+1)(D_n-D_n^*). \quad (5.10)$$

By solving (2.10) and (2.11) for the incident sound wave only (i.e. when the sphere is absent) it can be readily derived that

$$\langle \boldsymbol{v}_I^{(2)} \rangle = -\frac{|kA|^2}{4\omega}(\boldsymbol{k}+\boldsymbol{k}^*)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}-\mathrm{i}\boldsymbol{k}^*\cdot\boldsymbol{r}). \tag{5.11}$$

Substitution of (5.11) into (5.7) yields

$$F_D = -\tfrac{3}{2}\mathrm{i}\pi\rho_0 AA^* xx^*(x+x^*)\,x_v^{-2}\frac{\sin(x-x^*)}{x-x^*}. \tag{5.12}$$

The scalar velocity potential of the plane standing wave is given by

$$\phi_I = A\cos(\boldsymbol{k}\cdot\boldsymbol{r}+kd)\exp(-\mathrm{i}\omega t), \tag{5.13}$$

where d is the distance between the equilibrium centre of the sphere and the nearest plane of the velocity nodes. By expanding (5.13) in Legendre polynomials and comparing the series obtained with (3.3), one finds

$$A_n = \tfrac{1}{2}A(2n+1)\,\mathrm{i}^n[\exp(\mathrm{i}kd)+(-1)^n\exp(-\mathrm{i}kd)]. \tag{5.14}$$

Substituting (5.14) into (5.5), one obtains an expression for the radiation pressure exerted by the plane standing wave:

$$F_R = \tfrac{3}{4}\pi\rho_0 AA^*\sum_{n=0}^{\infty}(-1)^n(n+1)(D_n\sin(2kd)+D_n^*\sin(2k^*d)). \tag{5.15}$$

It can be easily verified that for the case of the standing wave

$$\langle \boldsymbol{v}_I^{(2)} \rangle = \frac{\mathrm{i}|kA|^2}{8\omega}[(\boldsymbol{k}-\boldsymbol{k}^*)\sin[(k+k^*)(z+d)]-(\boldsymbol{k}+\boldsymbol{k}^*)\sin[(k-k^*)(z+d)]]. \tag{5.16}$$

Substitution of (5.16) into (5.7) results in

$$F_D = -\tfrac{3}{4}\pi\rho_0 AA^*\frac{xx^*}{x_v^2}\left[(x-x^*)\sin(kd+k^*d)\frac{\sin(x+x^*)}{x+x^*} - (x+x^*)\sin(kd-k^*d)\frac{\sin(x-x^*)}{x-x^*}\right]. \tag{5.17}$$

Next, some limiting cases of interest are considered. It is to be emphasized that the force F_D (which, generally speaking, does not concern the radiation pressure) will only be mentioned in the cases when it is of the same order as the force F_R.

6. Radiation pressure on the sphere in the limiting case $|x| \ll 1 \ll |x_v|$, $|\tilde{x}| \ll 1 \ll |\tilde{x}_v|$

The approximate expressions for the functions S_{jn} required for calculating the radiation pressure in this limit are given in §C.2 of Appendix C.

6.1. $\lambda = (\rho_0 c^2)/(\tilde{\rho}_0 \tilde{c}^2) = O(1)$, $\mu = [(\rho_0\eta)/(\tilde{\rho}_0\tilde{\eta})]^{\frac{1}{2}} = O(1)$ *or* $\lambda \ll 1$, $\mu \ll 1$

These conditions describe the motion of a liquid drop suspended in another liquid or in a gas. The principal contribution to (5.5) comes from D_0 and D_1. The approximate expressions for the constants α_0, α_1 and β_1 required for calculating these terms are given in §B.1.1 of Appendix B.

6.1.1. *Plane progressive wave*

Substituting all the necessary quantities from the above-mentioned Appendices into (5.10) and retaining only the leading term in the expansion of the radiation pressure in powers of x, one obtains

$$F_R = 6\pi\rho_0 AA^* \frac{(\tilde{\rho}_0-\rho_0)^2}{(2\tilde{\rho}_0+\rho_0)^2(1+\mu)}(k_0 R)^3 \frac{\delta}{R}, \tag{6.1}$$

where $k_0 = \omega/c$.

In a perfect fluid, the radiation pressure exerted by the plane progressive wave is known to be proportional to $(k_0 R)^6$ (Yosioka & Kawasima 1955). It follows from (6.1) that the leading term in the expression for the radiation force in a viscous fluid is proportional to $(k_0 R)^3 \delta/R$. Thus, the viscosity, even though it is small, can cause a substantial increase of the radiation pressure. This phenomenon must be observed for a drop under the condition $R \ll \lambda_s(\delta/\lambda_s)^{\frac{1}{4}}$.

6.1.2. *Plane standing wave*

The calculations show that, in this case, the leading term in the expression for the radiation pressure in a viscous fluid is the same as that for the radiation pressure in a perfect fluid.

6.2. $\lambda \gg 1$, $\mu \gg 1$

These conditions describe the motion of a gas bubble immersed in a liquid. The principal contribution to (5.5) comes from D_0. The approximate expressions for the constants α_0, α_1 and β_1 required for calculating this term are given in §B.1.2 of Appendix B.

6.2.1. *Plane progressive wave*

Substituting all the necessary quantities from the above mentioned Appendices into (5.10), one obtains

$$F_R = 2\pi\rho_0 AA^* \frac{(k_0 R)^2 + \frac{1}{2}(7-3\omega_0^2/\omega^2)(k_0 R)(\delta/R)^2}{(1-\omega_0^2/\omega^2)^2 + [k_0 R + 2(\delta/R)^2]^2}, \tag{6.2}$$

where ω_0 is the angular resonance frequency of the gas bubble which is given by

$$\omega_0 = \frac{1}{R}(3\tilde{c}^2\tilde{\rho}_0/\rho_0 - 2\sigma/\rho_0 R)^{\frac{1}{2}}. \tag{6.3}$$

By comparing (6.2) with the corresponding formula for the radiation pressure in a perfect fluid (see Yosioka & Kawasima 1955), we conclude that taking into account the viscosity yields a correction, the magnitude of which depends on the ratio of the quantities $k_0 R$ and R/δ. This correction becomes dominant when $R \ll (\lambda_S \delta)^{\frac{1}{2}}$. It is easily seen that since $\lambda_s \gg R \gg \delta$, as is assumed, this situation, in principle, can occur. Moreover, if this happens, the force F_R can change its sign if $\omega_0^2 > \frac{7}{3}\omega^2$, i.e. the bubble will move in the direction of the sound-emitting transducer, while in a perfect fluid the bubble always moves in the direction of wave propagation (Yosioka & Kawasima 1955).

6.2.2. *Plane standing wave*

In this case, the leading term in the expression for the radiation pressure in a viscous fluid is identical with that for the radiation pressure in a perfect fluid.

7. Radiation pressure on the sphere in the limiting case $|x| \ll |x_v| \ll 1$, $|\tilde{x}| \ll |\tilde{x}_v| \ll 1$

The approximate expressions for the functions S_{jn} required for calculating the radiation pressure in this limit are given in §C.3 of Appendix C.

7.1. $\lambda = O(1)$, $\rho_0/\tilde{\rho}_0 = O(1)$ *or* $\lambda \ll 1$, $\rho_0 \ll \tilde{\rho}_0$, *the ratio* $\eta/\tilde{\eta}$ *may be arbitrary*

This is the case of a liquid drop suspended in another liquid or in a gas. The principal contribution to (5.5) comes from D_0 and D_1. The approximate expressions for the constants α_n and β_n ($n = 0, 1, 2$) required for calculating these terms are given in §B.2.1 of Appendix B.

7.1.1. *Plane progressive wave*

Substituting all the necessary quantities from the above mentioned Appendices into (5.10) and retaining only the leading term, one finds

$$F_R = \tfrac{1}{6}\pi\rho_0 AA^*(5-2\lambda-8f_1)(k_0 R)^3(\delta/R)^2. \tag{7.1}$$

The expression for the function f_1 is given in §B.2.1 of Appendix B. It should also be noted that (7.1) is valid under the condition

$$g_6 \gg \frac{40\sigma}{\rho_0\omega^2 R^3}(1+\eta/\tilde{\eta})|x_v^4|, \tag{7.2}$$

where

$$g_6 = 89+48\eta/\tilde{\eta}+38\tilde{\eta}/\eta \tag{7.3}$$

(see the note at the end of §B.2.1 of Appendix B).

As already mentioned, the radiation pressure exerted by the plane progressive wave is proportional to $(k_0 R)^6$ in a perfect fluid. It follows from (7.1) that the force F_R is proportional to $(k_0 R)^3(\delta/R)^2$, the quantity δ/R being large. This leads us to conclude that the magnitude of the radiation force in a viscous fluid is substantially larger than that of the radiation force in a perfect fluid. Moreover, it is easily seen that the expression between the first round brackets in (7.1) can change its sign. That is to say, the force given by (7.1) can set the drop moving both in the direction of wave propagation and in the opposite direction, while in a perfect fluid the drop is always urged away in the direction of wave propagation.

These conclusions may be illustrated by considering the particular case of an ethyl alcohol drop in glycerin. For glycerin at 20 °C and $p_0 = 10^5$ Pa one has $\rho_0 = 1.26 \times 10^3$ kg m^{-3}, $c = 1.9 \times 10^3$ m s^{-1}, $\eta = 1.48$ Pa s. For ethyl alcohol the same quantities are $\tilde{\rho}_0 = 0.79 \times 10^3$ kg m^{-3}, $\tilde{c} = 1.04 \times 10^3$ m s^{-1}, $\tilde{\eta} = 1.2 \times 10^{-3}$ Pa s, $\sigma = 2.23 \times 10^{-2}$ N m^{-1}. Setting $R = 10^{-6}$ m and $f = 1$ kHz (where f is the sound wave frequency), one obtains $|x| \approx 3.3 \times 10^{-6}$, $|x_v| \approx 2.3 \times 10^{-3}$, $|\tilde{x}| \approx 6.10^{-6}$, $|\tilde{x}_v| \approx 6.4 \times 10^{-2}$. Thus, the conditions $|x| \ll |x_v| \ll 1$ and $|\tilde{x}| \ll |\tilde{x}_v| \ll 1$ are satisfied. It can be verified that condition (7.2) is satisfied, too. Substituting all the necessary quantities into (7.1), one finds

$$F_{Rv} \approx -\pi\rho_0 AA^*(k_0 R)^3 \times 2.9 \times 10^5. \tag{7.4}$$

The radiation pressure calculated by the corresponding formula for a perfect fluid (see Yosioka & Kawasima 1955) is given by

$$F_{Rp} \approx \pi\rho_0 AA^*(k_0 R)^3 \times 1.6 \times 10^{-16}. \tag{7.5}$$

The ratio of $|F_{Rv}|$ to $|F_{Rp}|$ is of the order of 10^{21}.

It is to be noted that the force F_D given by (5.12) is of the same order as the force F_R in the case involved. Actually, passing to the limit $|x| \ll 1$, from (5.12) one obtains

$$F_D = -\tfrac{3}{2}\pi\rho_0 AA^*(k_0 R)^3(\delta/R)^2. \tag{7.6}$$

It follows that the total force F is written as

$$F = -\tfrac{1}{3}\pi\rho_0 AA^*(2+\lambda+4f_1)(k_0 R)^3(\delta/R)^2. \tag{7.7}$$

For a drop placed in a gas the total force F can become very small since $\lambda \ll 1$ and $f_1 \to -\frac{1}{2}$ as $\eta/\tilde{\eta} \to 0$. In this case, the next term in the expansion of F_R in powers of R/δ should be found since it may be of the same order as that given by (7.7).

7.1.2. *Plane standing wave*

Substituting all the necessary quantities from the above mentioned Appendices into (5.15) and retaining only the leading term, one obtains

$$F_R = \pi\rho_0 AA^* \sin(2k_0 d)(k_0 R)^3 G, \tag{7.8}$$

where

$$G = \tfrac{1}{3}(S_1+S_2+S_3), \tag{7.9}$$

$$S_1 = 1-\lambda+\tfrac{9}{10}(1-\tilde{\rho}_0/\rho_0), \tag{7.10}$$

$$S_2 = 2f_1(2f_2-1)-4f_4, \tag{7.11}$$

$$S_3 = \frac{\tilde{\rho}_0-\rho_0}{30\rho_0(3+2\eta/\tilde{\eta})}(43-10\lambda-120(3-\eta/\tilde{\eta})(f_3-f_4)+50f_1). \tag{7.12}$$

The expressions for the functions f_j are given in §B.2.1 of Appendix B. Just as for (7.1), formula (7.8) is valid as long as condition (7.2) holds.

As before, let us study (7.8) by comparing it with the corresponding expression for a perfect fluid. The latter is known to be proportional to $(k_0 R)^3$. The radiation force in a viscous fluid is found to be proportional to $(k_0 R)^3$, too, but its structure is quite different. To demonstrate the distinctions, let us consider two limiting cases: $\tilde{\eta} \gg \eta$ and $\tilde{\eta} \ll \eta$. For the first of these, from (7.9)–(7.12) one obtains

$$G = \tfrac{1}{3}(1+\chi_1^2)^{-1}\left\{1-\lambda+\chi_1^2(2-\lambda)-\frac{\tilde{\rho}_0-\rho_0}{90\rho_0}(5(9+2\lambda)+\chi_1^2(63+10\lambda))\right\}, \tag{7.13}$$

where

$$\chi_1 = \frac{20\sigma}{19\omega R\tilde{\eta}}, \tag{7.14}$$

and instead of (7.2) one has

$$\sigma R\rho_0/(\eta\tilde{\eta}) \ll 1. \tag{7.15}$$

Formulae (7.13)–(7.15) describe the motion of a liquid drop in a gas or of a high-viscosity liquid drop in a low-viscosity liquid. It is apparent that the factor G can be both positive and negative. A drop for which $G > 0$ is urged away towards the velocity antinodes. Conversely, a drop for which $G < 0$ is forced away towards the velocity nodes. The same phenomenon occurs in a perfect fluid, too, but the condition under which the radiation force changes its sign is different. To illustrate this, let us consider the particular case of a glycerin drop in an air. The data for glycerin are given above, except σ that is taken to be equal to 6.57×10^{-2} N m^{-1}. For air under the same conditions one has $\rho_0 = 1.2$ kg m^{-3}, $c = 330$ m s^{-1}, $\eta = 1.8\times10^{-5}$ Pa s. Setting $R = 10^{-6}$ m and $f = 1$ kHz, one obtains: $|x| \approx 1.9\times10^{-5}$, $|x_v| \approx 2\times10^{-2}$, $|\tilde{x}| \approx$

3.3×10^{-6}, $|\tilde{x}_v| \approx 2.3 \times 10^{-3}$. Note that the conditions $|x| \ll |x_v| \ll 1$ and $|\tilde{x}| \ll |\tilde{x}_v| \ll 1$ are satisfied, as well as (7.15). Substituting all the necessary quantities into (7.8), (7.13) and (7.14), one finds

$$F_{Rv} \approx -\pi\rho_0 AA^* \sin(2k_0 d)(k_0 R)^3 \times 2.4 \times 10^2, \tag{7.16}$$

while for a perfect fluid we have

$$F_{Rp} \approx \pi\rho_0 AA^* \sin(2k_0 d)(k_0 R)^3 \times 0.22. \tag{7.17}$$

Thus, taking into account the viscosity causes a substantial increase of the radiation force and changes its sign.

Now let us consider the limit $\tilde{\eta} \ll \eta$. From (7.9)–(7.12), one obtains

$$G = \frac{19\rho_0 - 9\tilde{\rho}_0}{30\rho_0} - \frac{\lambda}{3} + \frac{2}{9(1+\chi_2^2)}\left[\frac{8\sigma}{3\rho_0\omega^2 R^3} - \frac{5-3\chi_2^2}{5}\right], \tag{7.18}$$

where

$$\chi_2 = \frac{5\sigma}{6\omega R\eta}, \tag{7.19}$$

and condition (7.2) becomes

$$\sigma R\rho_0/\eta^2 \ll 1. \tag{7.20}$$

This is the case of a low-viscosity liquid drop suspended in a high-viscosity liquid. To illustrate this, let us consider the particular case of a water drop in glycerin. The data for glycerin are given above. For water one has $\tilde{\rho}_0 = 10^3$ kg m^{-3}, $\tilde{c} = 1.5 \times 10^3$ m s^{-1}, $\tilde{\eta} = 10^{-3}$ Pa s, $\sigma = 7.27 \times 10^{-2}$ N m^{-1}. Setting $R = 10^{-6}$ m and $f = 1$ kHz as before, one obtains: $|x| \approx 3.3 \times 10^{-6}$, $|x_v| \approx 2.3 \times 10^{-3}$, $|\tilde{x}| \approx 4.2 \times 10^{-6}$, $|\tilde{x}_v| \approx 8 \times 10^{-2}$. It is apparent that the conditions $|x| \ll |x_v| \ll 1$ and $|\tilde{x}| \ll |\tilde{x}_v| \ll 1$ are satisfied. It is easily verified that (7.20) is satisfied, too. Substituting all the necessary quantities into (7.8) and (7.18), one finds

$$F_{Rv} \approx \pi\rho_0 AA^* \sin(2k_0 d)(k_0 R)^3 \times 1.99 \times 10^4. \tag{7.21}$$

Meanwhile, for a perfect fluid the radiation force is given by

$$F_{Rp} \approx -\pi\rho_0 AA^* \sin(2k_0 d)(k_0 R)^3 \times 0.46. \tag{7.22}$$

By comparing (7.21) with (7.22), we conclude that the same distinctions occur as in the first case, i.e. the radiation force exerted by a plane standing wave in a viscous fluid is really drastically different from that due to a plane standing wave in a perfect fluid.

7.2. $\lambda \gg 1$, $\rho_0 \gg \tilde{\rho}_0$, $\eta \gg \tilde{\eta}$

In this case, a gas bubble immersed in a liquid, the principal contribution to (5.5) comes from D_0. The approximate expressions for the constants α_0, α_1 and β_1 required for calculating this term are given in §B.2.2 of Appendix B.

7.2.1. *Plane progressive wave*

Substituting all the necessary quantities from the above-mentioned Appendices into (5.10) and retaining only the leading term, one obtains

$$F_R = \pi\rho_0 AA^* \frac{(5-\omega_0^2/\omega^2)(k_0 R)(\delta/R)^2}{(1-\omega_0^2/\omega^2)^2 + 4(\delta/R)^4}. \tag{7.23}$$

The radiation pressure exerted by a plane progressive wave on a gas bubble in a perfect fluid is known to be proportional to $(k_0 R)^2$ (Yosioka & Kawasima 1955). It is apparent that the radiation force given by (7.23) is substantially larger than that in a perfect fluid. Moreover, the force F_R can change its sign according to whether $\omega_0^2 < 5\omega^2$ or $\omega_0^2 > 5\omega^2$. Thus, for gas bubbles the same phenomena occur as for drops.

To illustrate this, let us consider the particular case of an air bubble in glycerin. All the required data for glycerin and air have been given above. Setting $R = 10^{-6}$ m and $f = 1$ kHz, from (7.23) one finds

$$F_{Rv} \approx -\pi\rho_0 AA^* \times 2.2 \times 10^{-7}, \tag{7.24}$$

while for a perfect fluid we have

$$F_{Rp} \approx \pi\rho_0 AA^* \times 8 \times 10^{-25}. \tag{7.25}$$

The ratio of $|F_{Rv}|$ to $|F_{Rp}|$ is of the order of 10^{17}.

7.2.2. *Plane standing wave*

Substituting all the necessary quantities from the corresponding Appendices into (5.15) and retaining only the leading term in the expansion of the radiation pressure in powers of $(k_0 R)$, one obtains

$$F_R = \pi\rho_0 AA^* \sin(2k_0 d)(k_0 R)\frac{1-\omega_0^2/\omega^2-(\delta/R)^4}{(1-\omega_0^2/\omega^2)^2+4(\delta/R)^4}. \tag{7.26}$$

By comparing (7.26) with the corresponding formula for a perfect fluid (Yosioka & Kawasima 1955), we conclude that the radiation force in a viscous fluid can be in general substantially larger than that in a perfect fluid. To illustrate this, let us again consider the case of an air bubble in glycerin. Then for the ratio of F_{Rv} to F_{Rp}, one obtains

$$F_{Rv}/F_{Rp} \approx 2.6 \times 10^4. \tag{7.27}$$

It follows that the radiation force exerted by a standing wave is increasing in a viscous fluid, although not so much as that due to a progressive wave, the former having the same sign as in a perfect fluid.

To summarize, the calculation of the radiation pressure from the formulae for a perfect fluid in cases when the viscosity effect is not negligible gives, both quantitatively and qualitatively, wrong results.

Appendix A. Elements of the matrix $\boldsymbol{M}$ ($n \geqslant 1$)

$$m_{11} = -xh_n'(x), \quad m_{12} = -n(n+1)h_n(x_v), \quad m_{13} = \tilde{x}j_n'(\tilde{x}), \quad m_{14} = n(n+1)j_n(\tilde{x}_v),$$

$$m_{21} = -h_n(x), \quad m_{22} = -h_n(x_v) - x_v h_n'(x_v), \quad m_{23} = j_n(\tilde{x}), \quad m_{24} = \tilde{x}_v j_n'(\tilde{x}_v) + j_n(\tilde{x}_v),$$

$$m_{31} = m_{21}, \quad m_{32} = 0,$$

$$m_{33} = \frac{\tilde{\rho}_0}{\rho_0}\left[j_n(\tilde{x}) - \frac{2(1-\eta/\tilde{\eta})}{\tilde{x}_v^2}(n(n-1)j_n(\tilde{x}) + 2\tilde{x}j_{n+1}(\tilde{x})) - \frac{(n-1)(n+2)}{\tilde{\rho}_0\omega^2 R^3}\sigma\tilde{x}j_n'(\tilde{x})\right],$$

$$m_{34} = n(n+1)\frac{\tilde{\rho}_0}{\rho_0}\left[\frac{2(1-\eta/\tilde{\eta})}{\tilde{x}_v^2}((1-n)j_n(\tilde{x}_v) + \tilde{x}_v j_{n+1}(\tilde{x}_v)) - \frac{(n-1)(n+2)}{\tilde{\rho}_0\omega^2 R^3}\sigma j_n(\tilde{x}_v)\right],$$

$$m_{41} = 0, \quad m_{42} = -h_n(x_v), \quad m_{43} = \frac{2\tilde{\rho}_0(1-\eta/\tilde{\eta})}{\rho_0\tilde{x}_v^2}((1-n)j_n(\tilde{x}) + \tilde{x}j_{n+1}(\tilde{x})),$$

$$m_{44} = \frac{\tilde{\rho}_0}{\rho_0}\left[j_n(\tilde{x}_v) - \frac{2(1-\eta/\tilde{\eta})}{\tilde{x}_v^2}((n^2-1)j_n(\tilde{x}_v) + \tilde{x}_v j_{n+1}(\tilde{x}_v))\right].$$

Appendix B. Approximate expressions for the constants α_n and β_n ($n = 0, 1, 2$) in two limiting cases

B.1. *Case* $|x| \ll 1 \ll |x_v|$, $|\tilde{x}| \ll 1 \ll |\tilde{x}_v|$

B.1.1. $\lambda = (\rho_0 c^2)/(\tilde{\rho}_0 \tilde{c}^2) = O(1)$, $\mu = [(\rho_0 \eta)/(\tilde{\rho}_0 \tilde{\eta})]^{\frac{1}{2}} = O(1)$ *or* $\lambda \ll 1$, $\mu \ll 1$

$$\alpha_0 = -\tfrac{1}{3}\mathrm{i}x^3(1-\lambda), \quad \alpha_1 = \frac{\mathrm{i}x^3(\tilde{\rho}_0-\rho_0)}{3(2\tilde{\rho}_0+\rho_0)}\left(1+\frac{6\mathrm{i}(\tilde{\rho}_0-\rho_0)}{x_v(2\tilde{\rho}_0+\rho_0)(1+\mu)}\right),$$

$$\beta_1 = \frac{\mathrm{i}x(\rho_0-\tilde{\rho}_0)\exp(-\mathrm{i}x_v)}{(2\tilde{\rho}_0+\rho_0)(1+\mu)}\left(1-\mathrm{i}\frac{9\rho_0+(2\tilde{\rho}_0+\rho_0)\mu(1-2\mu)}{x_v(2\tilde{\rho}_0+\rho_0)(1+\mu)}\right).$$

B.1.2. $\lambda \gg 1$, $\mu \gg 1$

$$\alpha_0 = \frac{\mathrm{i}x(\omega_0^2/\omega^2-1+\mathrm{i}x-4/x_v^2)}{(\omega_0^2/\omega^2-1)^2+(x+4\mathrm{i}/x_v^2)^2}, \quad \omega_0^2 = \frac{3\tilde{c}^2\tilde{\rho}_0}{R^2\rho_0}\left(1-\frac{2\sigma}{3\tilde{c}^2\tilde{\rho}_0 R}\right),$$

$$\alpha_1 = \frac{\mathrm{i}x^3(\tilde{\rho}_0-\rho_0)}{3(2\tilde{\rho}_0+\rho_0)}\left(1+\frac{6\mathrm{i}\tilde{\rho}_0(\tilde{\rho}_0-\rho_0)}{\tilde{x}_v\rho_0(2\tilde{\rho}_0+\rho_0)}\right), \quad \beta_1 = \frac{\mathrm{i}x(\rho_0-\tilde{\rho}_0)\exp(-\mathrm{i}x_v)}{\mu(2\tilde{\rho}_0+\rho_0)}\left(1+\frac{2\mathrm{i}\mu}{x_v}\right).$$

B.2. *Case* $|x| \ll |x_v| \ll 1$, $|\tilde{x}| \ll |\tilde{x}_v| \ll 1$

B.2.1. $\lambda = O(1)$, $\rho_0/\tilde{\rho}_0 = O(1)$ *or* $\lambda \ll 1$, $\rho_0/\tilde{\rho}_0 \ll 1$, *the ratio of* η *to* $\tilde{\eta}$ *may be arbitrary*

$$\alpha_0 = -\tfrac{1}{3}\mathrm{i}x^3(1-\lambda), \quad \alpha_1 = \frac{\mathrm{i}x^3(\tilde{\rho}_0-\rho_0)}{9\rho_0}\left(1+\frac{2x_v^2(\tilde{\rho}_0-\rho_0)(1+\eta/\tilde{\eta})}{3\rho_0(3+2\eta/\tilde{\eta})}\right),$$

$$\beta_1 = \frac{\mathrm{i}xx_v^2(\tilde{\rho}_0-\rho_0)}{9\rho_0}\left[1-\frac{x_v^2(7+4\eta/\tilde{\eta}-4(1+\eta/\tilde{\eta})\tilde{\rho}_0/\rho_0)}{6(3+2\eta/\tilde{\eta})}\right],$$

$$\alpha_2 = \frac{4\mathrm{i}x^5}{9x_v^2}(f_1+x_v^2(f_3-f_1f_2)), \quad \beta_2 = \frac{2\mathrm{i}x^2x_v}{9}(f_1+x_v^2(f_4-f_1f_2)).$$

Here

$$f_1 = g_1/g_7, \quad f_2 = g_2g_5/2g_1, \quad f_3 = g_3g_6/10g_7, \quad f_4 = g_4g_6/2g_7,$$

$$g_1 = g_5g_6-|x_v^4|(1+\eta/\tilde{\eta})(5+2\eta/\tilde{\eta})160\sigma^2/(\rho_0^2\omega^4R^6),$$

$$g_2 = 19+38\tilde{\eta}/\eta-12\eta/\tilde{\eta}-\frac{\nu}{9\tilde{\nu}}(209+148\tilde{\eta}/\eta+48\eta/\tilde{\eta})+\frac{80\sigma}{\rho_0\omega^2R^3}(1+\eta/\tilde{\eta}),$$

$$g_3 = \frac{\nu}{9\tilde{\nu}}(25+370\tilde{\eta}/\eta-80\eta/\tilde{\eta})-12-19\tilde{\eta}/\eta-4\eta/\tilde{\eta}-\frac{40\sigma}{\rho_0\omega^2R^3}(5+2\eta/\tilde{\eta}),$$

$$g_4 = \frac{\nu}{9\tilde{\nu}}(5+74\tilde{\eta}/\eta-16\eta/\tilde{\eta})-3-4\eta/\tilde{\eta}-\frac{8\sigma}{\rho_0\omega^2R^3}(5+2\eta/\tilde{\eta}),$$

$$g_5 = (1-\tilde{\eta}/\eta)(19+16\eta/\tilde{\eta}), \quad g_6 = 89+48\eta/\tilde{\eta}+38\tilde{\eta}/\eta,$$

$$g_7 = g_6^2+|x_v^4|(1+\eta/\tilde{\eta})^2(40\sigma/\rho_0\omega^2R^3)^2.$$

Note that the expressions for α_2 and β_2 have been calculated under the assumption that $g_6 \gg (40\sigma/\rho_0\omega^2R^3)(1+\eta/\tilde{\eta})|x_v^4|$.

B.2.2. $\lambda \gg 1,\ \rho_0/\tilde{\rho}_0 \gg 1,\ \eta/\tilde{\eta} \gg 1$

$$\alpha_0 = \frac{\mathrm{i}x(\omega_0^2/\omega^2 - 1 - 4/x_v^2)}{(\omega_0^2/\omega^2-1)^2 - 16/x_v^4},\quad \alpha_1 = -\mathrm{i}x^3(3-x_v^2)/27,\quad \beta_1 = -\mathrm{i}xx_v^2(3-x_v^2)/27.$$

Appendix C. Expressions for the functions S_{jn} $(j = 1, \ldots, 9)$

C.1. *Exact expressions for the functions* S_{jn}

$$\begin{aligned} S_{1n} = {} & \tfrac{1}{2}((n+2)\,x^2 - nx^{2*})\,(H^{(0)}_{nn+1}(x,x) - H^{(2)}_{nn+1}(x,x)) - xx^*(H^{(0)}_{n+1n}(x,x) - H^{(2)}_{n+1n}(x,x)) \\ & + ((x^2 - x^{2*})/x_v^2)\,(nx^* H^{(1)}_{nn}(x,x) + (n+2)\,xH^{(1)}_{n+1n+1}(x,x) - xx^* H^{(0)}_{n+1n}(x,x)) \\ & + (xx^*/x_v^2)\,G^{(2)}_n(x), \end{aligned}$$

$$\begin{aligned} S_{2n} = {} & n(n+2)\{x_v x_v^*\,(H^{(0)}_{n+1n}(x_v,x_v) - H^{(2)}_{n+1n}(x_v,x_v)) - (n+1)\,x_v^2(H^{(0)}_{nn+1}(x_v,x_v) \\ & - H^{(2)}_{nn+1}(x_v,x_v)) - (n+1)\,[h^{(1)}_n(x_v)\,(h^{(2)}_n(x_v^*))' + (h^{(1)}_{n+1}(x_v))'\,h^{(2)}_{n+1}(x_v^*)]\}, \end{aligned}$$

$$\begin{aligned} S_{3n} = {} & (n+2)\Big\{\tfrac{1}{2}(nx_v^{2*} - (n+1)\,x^2)\,(H^{(0)}_{nn+1}(x,x_v) - H^{(2)}_{nn+1}(x,x_v)) - xx_v^*(H^{(0)}_{n+1n}(x,x_v) \\ & - H^{(2)}_{n+1n}(x,x_v)) + \frac{x^2x_v^*}{2}(H^{(-1)}_{nn}(x,x_v) - H^{(1)}_{nn}(x,x_v)) + \frac{xx_v^2}{2}(H^{(-1)}_{n+1n+1}(x,x_v) \\ & - H^{(1)}_{n+1n+1}(x,x_v)) + \frac{x^2}{x_v^2}(nx_v^*\,H^{(1)}_{nn}(x,x_v) - (n+1)\,xH^{(1)}_{n+1n+1}(x,x_v)) - L^{(1)}_n(x,x_v)\Big\}, \end{aligned}$$

$$\begin{aligned} S_{4n} = {} & n\Big\{\tfrac{1}{2}((n+2)\,x_v^2 - (n+1)\,x^{2*})\,(H^{(0)}_{nn+1}(x_v,x) - H^{(2)}_{nn+1}(x_v,x)) + x^*x_v(H^{(0)}_{n+1n}(x_v,x) \\ & - H^{(2)}_{n+1n}(x_v,x)) - \frac{x^*x_v^2}{2}(H^{(-1)}_{nn}(x_v,x) - H^{(1)}_{nn}(x_v,x)) + \frac{x^{2*}x_v}{2}(H^{(-1)}_{n+1n+1}(x_v,x) \\ & - H^{(1)}_{n+1n+1}(x_v,x)) - \frac{x^{2*}}{x_v^2}((n+1)\,x^*H^{(1)}_{nn}(x_v,x) - (n+2)\,x_v\,H^{(1)}_{n+1n+1}(x_v,x)) \\ & + K^{(2)}_n(x,x_v)\Big\}, \end{aligned}$$

$$\begin{aligned} S_{5n} = {} & \tfrac{1}{2}((n+2)\,x^2 - nx^{2*})\,(J^{(0)}_{nn+1}(x,x) - J^{(2)}_{nn+1}(x,x)) - xx^*(J^{(0)}_{n+1n}(x,x) - J^{(2)}_{n+1n}(x,x)) \\ & + \frac{x^2 - x^{2*}}{x_v^2}(nx^*J^{(1)}_{nn}(x,x) + (n+2)\,xJ^{(1)}_{n+1n+1}(x,x) - xx^*J^{(0)}_{n+1n}(x,x)) + \frac{xx^*}{2x_v^2}(G^{(1)}_n(x) \\ & + G^{(2)}_n(x)), \end{aligned}$$

$$\begin{aligned} S_{6n} = {} & \tfrac{1}{2}((n+2)\,x^2 - nx^{2*})\,(J^{(0)}_{n+1n}(x,x) - J^{(2)}_{n+1n}(x,x))^* - xx^*(J^{(0)}_{nn+1}(x,x) - J^{(2)}_{nn+1}(x,x))^* \\ & + \frac{x^2 - x^{2*}}{x_v^2}(nxJ^{(1)}_{nn}(x,x) + (n+2)\,x^*J^{(1)}_{n+1n+1}(x,x) - xx^*J^{(0)}_{nn+1}(x,x))^* \\ & + \frac{xx^*}{2x_v^2}(G^{(1)*}_n(x^*) + G^{(2)}_n(x)), \end{aligned}$$

$$S_{7n} = n\Big\{\tfrac{1}{2}((n+2)\,x_v^2-(n+1)\,x^{2*})\,(J^{(0)}_{n\,n+1}(x_v,x)-J^{(2)}_{n\,n+1}(x_v,x))+x^*x_v(J^{(0)}_{n+1\,n}(x_v,x)$$

$$-J^{(2)}_{n+1\,n}(x_v,x))-\frac{x^*x_v^2}{2}(J^{(-1)}_{nn}(x_v,x)-J^{(1)}_{nn}(x_v,x))+\frac{x^{2*}x_v}{2}(J^{(-1)}_{n+1\,n+1}(x_v,x)$$

$$-J^{(1)}_{n+1\,n+1}(x_v,x))-\frac{x^{2*}}{x_v^2}((n+1)\,x^*J^{(1)}_{nn}(x_v,x)-(n+2)\,x_v\,J^{(1)}_{n+1\,n+1}(x_v,x))$$

$$+\tfrac{1}{2}(K^{(1)}_n(x,x_v)+K^{(2)}_n(x,x_v))\Big\},$$

$$S_{8n} = (n+2)\Big\{\tfrac{1}{2}(nx_v^{2*}-(n+1)\,x^2)\,(J^{(0)}_{n+1\,n}(x_v,x)-J^{(2)}_{n+1\,n}(x_v,x))^*-xx_v^*\,(J^{(0)}_{n\,n+1}(x_v,x)$$

$$-J^{(2)}_{n\,n+1}(x_v,x))^*+\frac{x^2x_v^*}{2}(J^{(-1)}_{nn}(x_v,x)-J^{(1)}_{nn}(x_v,x))^*+\frac{xx_v^2}{2}(J^{(-1)}_{n+1\,n+1}(x_v,x)$$

$$-J^{(1)}_{n+1\,n+1}(x_v,x))^*+\frac{x^2}{x_v^2}(nx_v\,J^{(1)}_{nn}(x_v,x)-(n+1)\,x^*J^{(1)}_{n+1\,n+1}(x_v,x))^*-\tfrac{1}{2}(L^{(1)}_n(x,x_v)$$

$$+L^{(2)}_n(x,x_v))\Big\},$$

$$S_{9n} = \frac{xx^*}{x_v^2}(x^*j_n'(x)j_n'(x^*)+xj_{n+1}'(x)j_{n+1}'(x^*)).$$

Here

$$H^{(j)}_{nm}(x_1,x_2) = \int_1^\infty y^{-j}h^{(1)}_n(x_1y)\,h^{(2)}_m(x_2^*y)\,\mathrm{d}y$$

$$= \sum_{k=0}^{n}\sum_{q=0}^{m} B_{kq}\,x_1^{-(k+1)}(x_2^*)^{-(q+1)}E_M(\mathrm{i}x_2^*-\mathrm{i}x_1),$$

$$J^{(j)}_{nm}(x_1,x_2) = \int_1^\infty y^{-j}h^{(1)}_n(x_1y)\,j_m(x_2^*y)\,\mathrm{d}y$$

$$= \frac{1}{2}\sum_{k=0}^{n}\sum_{q=0}^{m} B_{kq}\,x_1^{-(k+1)}(x_2^*)^{-(q+1)}\,(E_M(\mathrm{i}x_2^*-\mathrm{i}x_1)-(-1)^{m+l}E_M(-\mathrm{i}x_2^*-\mathrm{i}x_1)),$$

$$B_{kq} = \frac{\mathrm{i}^{m-n}(-1)^k(n+k)!\,(m+q)!}{(2\mathrm{i})^{k+q}k!\,(n-k)!\,q!\,(m-q)!};\quad M = 2+k+q+j;\quad j=-1,0,1,2;$$

$$x = kR,\quad x_v = k_v\,R,$$

$$G^{(l)}_n(x) = x^*(h^{(1)}_n(x))'(h^{(l)}_n(x^*))'+x(h^{(1)}_{n+1}(x))'(h^{(l)}_{n+1}(x^*))',$$

$$L^{(l)}_n(x,x_v) = x(h^{(l)}_n(x))'(h^{(2)}_n(x_v^*))'-\frac{(n+1)\,x^2}{x_v^2}(h^{(l)}_{n+1}(x))'h^{(2)}_{n+1}(x_v^*),$$

$$K^{(l)}_n(x,x_v) = \frac{(n+1)\,x^{2*}}{x_v^2}(h^{(l)}_n(x^*))'h^{(1)}_n(x_v)-x^*(h^{(l)}_{n+1}(x^*))'(h^{(1)}_{n+1}(x_v))',$$

$$l=1,2;\quad (h^{(l)}_n(x))' = \mathrm{d}h^{(l)}_n(x)/\mathrm{d}x.$$

Here $h^{(l)}_n(z)$ is the spherical Hankel function of the lth kind, $E_n(z)$ is the integral exponent of order n (Abramowitz & Stegun 1965).

C.2. *Approximate expressions for the functions* $S_{jn}(n=0,1)$ *in the limiting case* $|x| \ll 1 \ll |x_v|$

$$S_{10} = 4x^{-3}x_v^{-2}, \qquad S_{11} = 81x^{-5}x_v^{-2},$$
$$S_{20} = 0, \qquad S_{21} = 3x_v^{-2}(1+2\mathrm{i})\exp(\mathrm{i}x_v - x_v),$$
$$S_{30} = 10x^{-1}x_v^{-2}\exp(-x_v), \qquad S_{31} = 30\mathrm{i}x^{-2}x_v^{-2}\exp(-x_v),$$
$$S_{40} = 0, \qquad S_{41} = 126x^{-3}x_v^{-2}\exp(\mathrm{i}x_v),$$
$$S_{50} = \tfrac{2}{3}\mathrm{i}(1+x_v^{-2}), \qquad S_{51} = \tfrac{2}{15}\mathrm{i}(5+14x_v^{-2}),$$
$$S_{60} = \tfrac{2}{3}i(1-x_v^{-2}), \qquad S_{61} = \tfrac{2}{15}\mathrm{i}(5-14x_v^{-2}),$$
$$S_{70} = 0, \qquad S_{71} = -\frac{2\mathrm{i}}{15}x^2x_v^{-2}\exp(\mathrm{i}x_v),$$
$$S_{80} = \frac{4\mathrm{i}}{3}x^2x_v^{-2}\exp(-x_v), \qquad S_{81} = 2xx_v^{-3}(1+x_v)\exp(-x_v),$$
$$S_{90} = S_{91} = \tfrac{1}{9}x^3x_v^{-2}.$$

C.3. *Approximate expressions for the functions* $S_{jn}(n=0,1)$ *in the limiting case* $|x| \ll |x_v| \ll 1$

The expressions for the functions S_{1n}, S_{5n}, S_{6n} and S_{9n} $(n=0,1)$ are identical with the ones given above.

$$S_{20} = 0, \qquad S_{21} = -6\mathrm{i}x_v^{-7}(27-x_v^2),$$
$$S_{30} = \tfrac{4}{3}x^{-1}x_v^{-4}(3+x_v^3), \qquad S_{31} = \mathrm{i}x^{-2}x_v^{-5}(162-15x_v^2),$$
$$S_{40} = 0, \qquad S_{41} = -\tfrac{9}{2}x^{-3}x_v^{-4}(18+x_v^2),$$
$$S_{70} = 0, \qquad S_{71} = -\tfrac{4}{15}\mathrm{i}x^2x_v^{-4}(7+4x_v^2),$$
$$S_{80} = -\tfrac{1}{3}\mathrm{i}x^2x_v^{-4}(2-5x_v^2), \qquad S_{81} = xx_v^{-3}(2-x_v^2).$$

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